PRACTICE SET FOR MIDTERM 2, SOLUTIONS

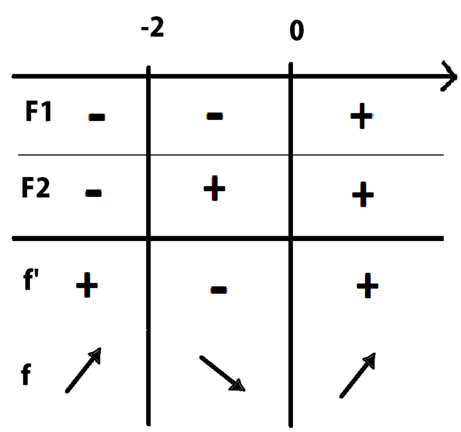
C) Do the following equations admit any real solutions? If so, how many?

1) $x^5 + \dfrac{1}{3}x^3 = 3 - e^x$

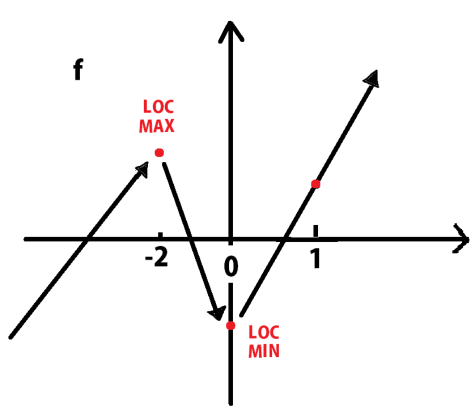
Rewrite the equation as $x^5 + \frac{1}{3}x^3 + e^x - 3 = 0$ and consider the function $f(x) = x^5 + \frac{1}{3}x^3 + e^x - 3 = 0$. This is a continuous function on $\mathbb{R}$ and we have $f(0) = 1 - 3 < 0$, $f(1) = 1 + 1/3 + e - 3 > 0$. Therefore by the IVT there is at least a solution in the interval [0,1].
Now $f'(x) = 5x^4 + x^2 + e^x$ which is always stricly positive. Hence, f is increasing in $\mathbb{R}$, and therefore the equation $f(x) = 0$ has only ONE real solution.

2) $2x^5 + 5x^4 - 3 = 0$

Consider the function $f(x) = 2x^5 + 5x^4 - 3$, which is continuous on $\mathbb{R}$. Note that $f(0) = -3 < 0$ and $f(1) = 4 > 0$ so by the IVT there is at least a solution in the interval $[0, 1]$.
$f'(x) = 10x^4 + 20x^3 = 10x^3(x + 2)$. We study the sign of f'.
$F_1 > 0 : 10x^3 > 0 \Rightarrow x > 0$
$F_2 > 0 : x + 2 > 0 \Rightarrow x > -2$



Therefore f is increasing in $]-\infty, -2[$, decreasing in $]-2, 0[$ and increasing in $]0, \infty[$. There is a local max at $x = -2$, and $f(-2) = -2^6 + 5 \cdot 2^4 - 3 = 2^4(-4 + 5) - 3 > 0$, and a local min at $x = 0$, and $f(0) = -3 < 0$. Since $\lim_{x \to -\infty} f(x) = -\infty$, combining all the above info we see that $f(x) = 0$ has 3 real solutions.



3∗) $\arctan(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{25}{12}$

We will show the equation has NO real solutions. Consider the functions

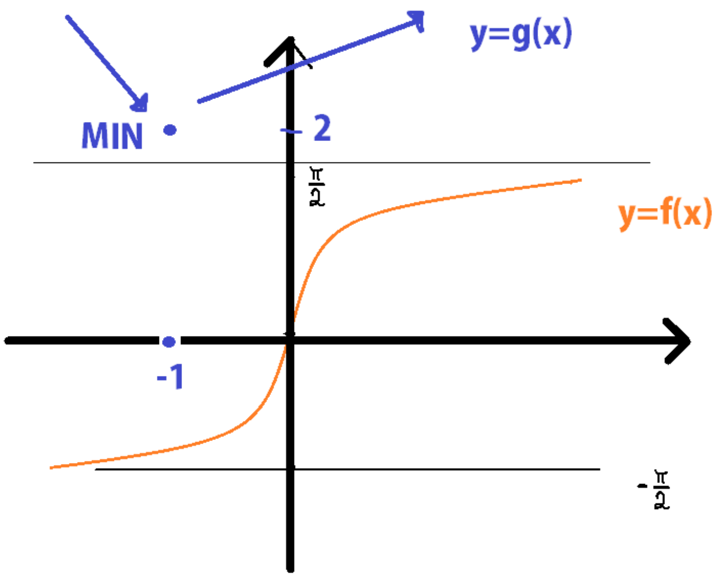
$$f(x) = \operatorname{arctg}(x) \quad \text{and} \quad g(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{25}{12}.$$

The natural domain of both of them is $\mathbb{R}$. The given equation has a solution if the graphs of f and g have intersection points. We know that

$$-\frac{\pi}{2} \leq \operatorname{arctg}(x) \leq \frac{\pi}{2}.$$

Now we study the monotonicity of g. $g'(x) = x^3 + x^2 = x^2(x+1)$. Since x^2 is always positive, $g'(x) > 0$ when $x > -1$.

So f has a GLOBAL MIN at $x = -1$, and $f(-1) = \frac{1}{4} - \frac{1}{3} + \frac{25}{12} = 2 > \frac{\pi}{2}$. This means that $g(x) \geq 2 > \operatorname{arctg}(x)$ for all $x \in \mathbb{R}$, and the curves can never intersect.



4*) $x^2 = \sin(x)$

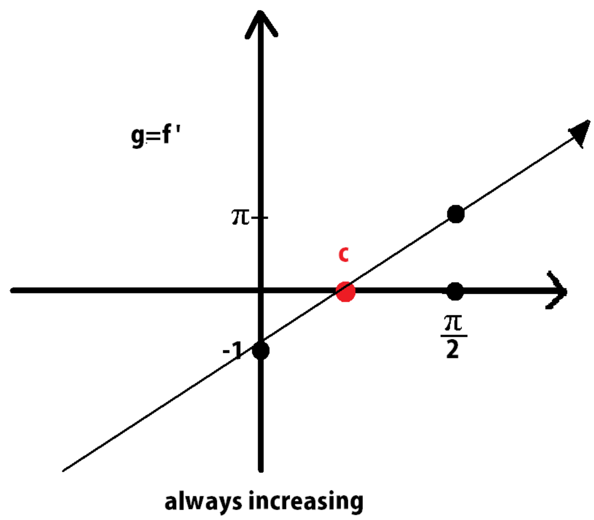
The equation is equivalent to $x^2 - \sin(x) = 0$. Consider the function

$$f(x) = x^2 - \sin(x)$$

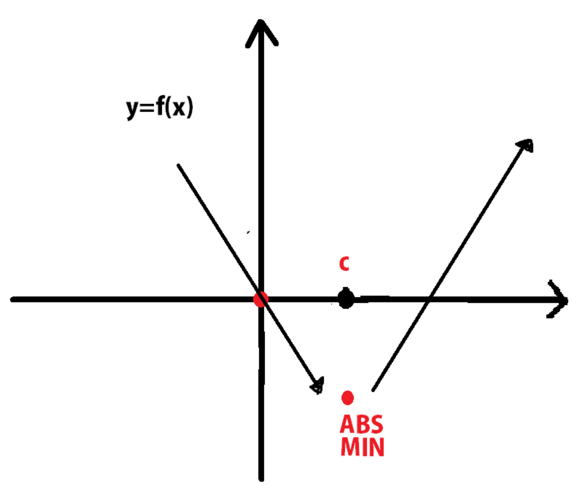
with natural domain $\mathbb{R}$. We readily see that $x = 0$ is a solution of the equation.
Now we study the monotonicity of f. $f'(x) = 2x - \cos(x)$. Unfortunately we cannot solve the inequality $2x - \cos(x) > 0$ explicitly. So we study the behaviour of the function

$$g(x) = f'(x) = 2x - \cos(x).$$

Note that g is continuous on $\mathbb{R}$ and $g(0) = -1 < 0$, $g(\pi/2) = \pi > 0$, so there exists $c \in [0, \pi/2]$ such that $g(c) = 0$ by the IVT. Also, $g'(x) = 2 + \sin(x)$ which is always positive. So g is always increasing, which means that $g(x) < 0$ for $x < c$ and $g(x) > 0$ for $x > c$.



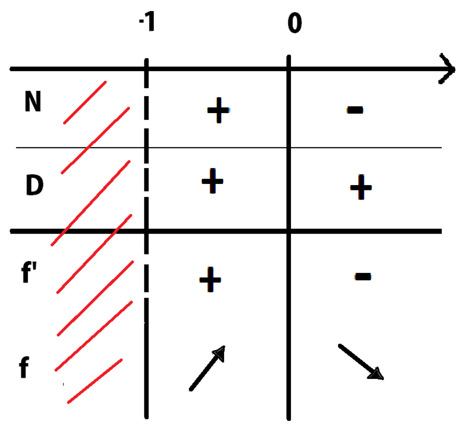
But $g = f'$, so this implies that $f(x)$ is decreasing for $x < c$ and increasing for $x > c$. Note that $\lim_{x \to +\infty} f(x) = +\infty$. Therefore the equation $f(x) = 0$ has 2 real solutions.



D) Prove that $\ln(x+1) \leq x$ for every $x > -1$.
This is equivalent to proving that $\ln(x+1) - x \leq 0$. Consider the function $f(x) = \ln(x+1) - x$ whose domain is $x > -1$. Let's study the monotonicity of this function. $f'(x) = \frac{1}{x+1} - 1 = \frac{-x}{x+1}$. Solve the inequality

$$\frac{-x}{x+1} > 0.$$

$$N > 0 : -x > 0 \Rightarrow x < 0$$
$$D > 0 : x+1 > 0 \Rightarrow x > -1$$



Therefore f is increasing in $]-1, 0[$ and decreasing in $]0, \infty[$, so it has a local (and global) max at $x = 0$. Since $f(0) = 0$, we get $f(x) \leq 0$ for all $x < -1$.

Alternative solution: consider the function $g(x) = \ln(x+1)$, with domain $x > -1$. Note that $g(0) = 0$ and $g'(0) = 1$ because $g'(x) = \frac{1}{x+1}$. Therefore the tangent line to g at the point $(0,0)$ is $y = x$. Since $g''(x) = \frac{-1}{(x+1)^2}$ which is always negative, g is concave down in its domain. By definition of concavity this means than g lies below all its tangent lines, in particular below the line $y = x$. This means $g(x) \leq x$.

E) Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that $f'(x) < 0 \quad \forall x \in \mathbb{R}$ and such that $\lim_{x \to -\infty} f(x) = +\infty$. Is it true that the equation $f(x) = 0$ has exactly one real solution?
It's FALSE. A counterexample is given by $f(x) = e^{-x}$. It satisfies all the requirements, but the equation $f(x) = 0$ has no real solutions.

F*) Can you find a differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that f(5)=5, f(-5)=-5 and $f'(x) \geq x^2 + 2$?
NO, such function doesn't exist. Indeed since f is differentiable everywhere, then it is also continuous everywhere. In particular f is contiuous in $[-5, 5]$ and differentiable in $]-5, 5[$ so the hypotheses of Lagrange's Mean Value Theorem apply. Therefore we conclude that there is $c \in [-5, 5]$ such that

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)} = 1.$$

But $f'(x)$ is always greater than 2, so this is impossible.

G*) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an even and differentiable function. Assuming the derivative is a continuous function, compute

$$\lim_{x \to 0} \frac{f(x) - f(0)}{\sin(x)}.$$

Since f is differentiable, it is continuous. So $\lim_{x \to 0}(f(x) - f(0)) = f(0) - f(0) = 0$. Also, $\lim_{x \to 0} \sin(x) = 0$ so we can try to apply L'Hospital's rule and compute the limit

$$\lim_{x \to 0} \frac{f'(x)}{\cos(x)} = \frac{f'(0)}{\cos(0)} = f'(0)$$

where we computed the limit using the fact that f' is continuous, so $\lim_{x \to 0} f'(x) = f'(0)$. Now, since f is EVEN, we have that f' is ODD. Indeed, differentiating the relation $f(-x) = f(x)$ we get by the chain rule

$$f'(-x) \cdot (-1) = f'(x)$$

which means precisely that f' is odd. But an odd function has to be 0 at $x = 0$, so $f'(0) = 0$. Therefore the result of the given limit is 0 by L'Hospital's rule.

H*) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$. Suppose that $f(0) = 1$, $f'(0) = 5$ and $f''(x) < 0$ for every $x \in \mathbb{R}$. Prove that $f(x) \leq 5x + 1$ for every $x \in \mathbb{R}$.

Since $f'' < 0$ for every $x \in \mathbb{R}$, f is concave down in $\mathbb{R}$. By definition this means that the graph of f lies below each tangent line. Since $f(0) = 1$ and $f'(0) = 5$, the tangent line at $x = 0$ is $y = 5x + 1$. Therefore $f(x) \leq 5x + 1$ for every $x \in \mathbb{R}$.